12/9/23 MATH4030 Lecture

Announcements:

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  Office hours by email appointment.
- No tutorial this week.

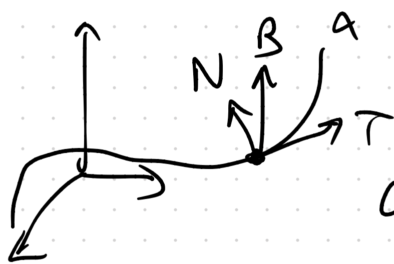
Frénet Formulas (in arc-length param.):

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

Curves Contd.

Goal: Discuss Fundamental theorem of Local Theory of Curves.

Recall: $\alpha : I \to \mathbb{R}^3$



$\{T, N, B\}$ O.N. frame (Frénet frame) moving along α.

$T = \frac{\alpha'}{|\alpha'|}$, $N = \frac{T'}{|T'|}$, $B = T \times N$.

arc-length param. s s.t. $|\alpha'(s)| \equiv 1$.

In arc-length param. s, $k(s)$ - curvature, $\tau(s)$ - torsion

$$T'(s) = k(s)N(s). \qquad B'(s) = -\tau(s)N(s).$$

Q: Want a general formula for $k(t)$, $\tau(t)$, with t an arbitrary param. (not neccessarily arc-length param.).

Prop: Given a regular curve $\alpha: I \to \mathbb{R}^3$

1) $k(t) = \dfrac{|\alpha' \times \alpha''|}{|\alpha'|^3}$ where $'$ is differentiation wrt. t.

2) $\tau(t) = \dfrac{\langle \alpha' \times \alpha'', \alpha''' \rangle}{|\alpha' \times \alpha''|^2}$

Pf: (Think: reduce back to arc-length param.).

Recall: $s(t) = \int_a^t |\alpha'(u)|\,du$, $\alpha(t) = \alpha(s(t))$. ← abusing notation.

(1): $\alpha' = \frac{d}{dt}\alpha(s(t)) = \alpha'(s(t))\frac{ds}{dt} = \frac{ds}{dt}T(s(t)) = |\alpha'|T(s(t))$.

$$T' = \frac{d}{dt} T(s(t)) = T'(s(t)) \frac{ds}{dt} = |\alpha'| k N.$$

$$\alpha'' = (|\alpha'| T)' = |\alpha'|' T + |\alpha'| T' = |\alpha'|' T + k|\alpha'|^2 N.$$

$$\alpha' \times \alpha'' = |\alpha'| T \times (|\alpha'|' T + k|\alpha'|^2 N)$$

$$= k|\alpha'|^3 \, T \times N = k|\alpha'|^3 B.$$

$$\Rightarrow k = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}, \quad (k \geq 0 \text{ by def'n}).$$

For (2), $\alpha''' = (|\alpha|' T + k|\alpha'|^2 N)' = |\alpha'|'' T + |\alpha'|' T' + k' |\alpha'|^2 N$
$+ k(|\alpha'|^2)' N + k|\alpha'|^2 N'$

By Frénet, $k|\alpha'|^2 N' = k|\alpha'|^2 \frac{ds}{dt} \frac{dN}{ds} = k|\alpha'|^3 (-kT + \tau B)$

So $\alpha''' = fT + gN + \tau k |\alpha'|^3 B$ where f, g are some differentiable functions in t.

$$\tau k|\alpha'|^3 = \langle \alpha''', B\rangle = \langle \alpha''', T\times N\rangle$$

By calculation in (1), $T\times N = T\times \left(\alpha'' - |\alpha'|' T\right)\cdot \frac{1}{k|\alpha'|^2}$

$$= \left(\frac{\alpha'}{|\alpha'|}\times\alpha''\right)\frac{1}{k|\alpha'|^2} = \frac{1}{k|\alpha'|^3}\left(\alpha'\times\alpha''\right).$$

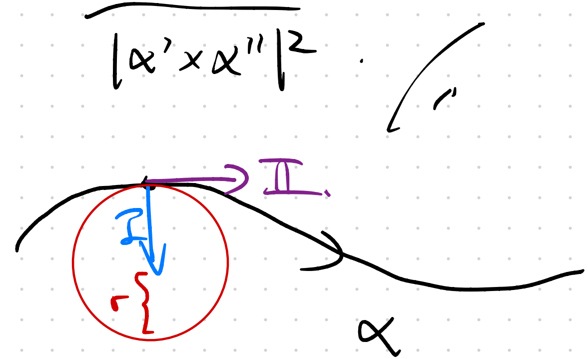
Note, by taking norms, $1=|B|^2=|T\times N|^2 = \left(\frac{1}{k|\alpha'|^3}\right)^2|\alpha'\times\alpha''|^2$

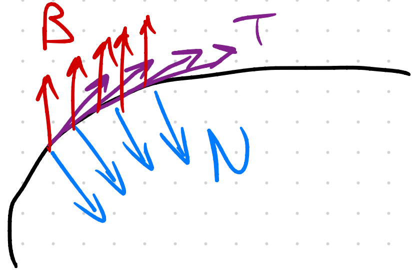
$$\Rightarrow |\alpha'\times\alpha''|^2 = \left(k|\alpha'|^3\right)^2.$$

$$\tau = \frac{\langle\alpha''', T\times N\rangle}{k|\alpha'|^3} = \frac{\langle \alpha''', \alpha'\times\alpha''\rangle}{\left(k|\alpha'|^3\right)^2} = \frac{\langle \alpha''', \alpha'\times\alpha''\rangle}{|\alpha'\times\alpha''|^2}.$$

Physical interpretation of α'':

$$\alpha'' = \underbrace{k|\alpha'|N}_{\text{angular acceleration}} + \underbrace{|\alpha'|'T}_{\text{linear acceleration}} = \mathrm{I} + \mathrm{II}$$





$r =$ radius of curvature $\triangleq \frac{1}{k}$
(radius of the circle corresponding to angular motion)

$\{T, N\}$ – called the osculating plane.

$\{T, N, B\}$ – Frénet frame:

$$\underbrace{\begin{bmatrix} T \\ N \\ B \end{bmatrix}'}_{1^{st}\text{ order}} = \underbrace{\begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}}_{\text{fn's.}} \underbrace{\begin{bmatrix} T \\ N \\ B \end{bmatrix}}_{0^{th}\text{ order}}$$

Geometric viewpoint: Transformation from a straight line L to $\alpha : I \to \mathbb{R}^3$

$L \to$ Bending $\to$ Twisting $\to \alpha$.
(Curvature) (Torsion)

Expect: a curve to be completely determined by $k(t)$, $\tau(t)$ (modulus some rigid motion in $\mathbb{R}^3$).

Rigid motions: rotation, translation.

"the same curve".

Fundamental Thm of Local Theory of Curves

Given differentiable fn's $k(s)>0$, $\tau(s)>0$ where $s\in I=[a,b]$, $\exists$ regular curve $\alpha: I\to\mathbb{R}^3$ which is param. by arc-length s.t.

(1) $k(s)$ = curvature of α at s

(2) $\tau(s)$ = torsion of α at s

Moreover, if $\bar{\alpha}: I\to\mathbb{R}^3$, arc-length param. curve with (1), (2) also, then

$\exists A \in SO(3)$ and $x_0 \in \mathbb{R}^3$ s.t. $\bar{\alpha} = A \cdot \alpha + x_0$.

(i.e. $\bar{\alpha} \cong \alpha$ up to a rigid motion).

where $SO(3) = \{ A \in M_{3\times3}(\mathbb{R}) : \det A > 0,\ A^T A = A A^T = Id \}$.

<u>Rmk</u>: (Easy part): Rigid motion preserves arc-length, curvature, torsion

geometric quantities that are independent of coordinates, "coordinate free".

<u>Pf</u>: $\bar{\alpha}(s) = A \cdot \alpha(s) + p$, $p \in \mathbb{R}^3$.

$\bar{\alpha}'(s) = A \alpha'(s)$

$$|\bar{\alpha}'(s)|^2 = \langle \bar{\alpha}', \bar{\alpha}' \rangle = \langle A \cdot \alpha', A \cdot \alpha' \rangle$$

Id. $(SO(3))$.

$$= (A\alpha')^T (A\alpha') = \alpha'^T \cancel{A^T A}\, \alpha'$$

$$= \langle \alpha', \alpha' \rangle = |\alpha'|^2.$$

So arc-length is preserved.

$$\bar{k}(s) = |\bar{\alpha}''(s)| = |A \cdot \alpha''| \underset{\substack{\uparrow \\ A \in SO(3), \\ \text{same computation} \\ \text{as above}}}{=} |\alpha''| = k(s).$$

$$\bar{\tau}(s) = \frac{\langle \bar{\alpha}' \times \bar{\alpha}'', \bar{\alpha}''' \rangle}{|\bar{\alpha}' \times \bar{\alpha}''|^2}$$

$$|\bar{\alpha}' \times \bar{\alpha}''|^2 \overset{\text{linear algebra}}{=} |\bar{\alpha}'|^2 |\bar{\alpha}''|^2 - |\langle \bar{\alpha}', \bar{\alpha}'' \rangle|^2$$

$$= |A\alpha'|^2 |A\alpha''|^2 - |\langle A\alpha', A\alpha'' \rangle|^2$$

$$\underset{A \in SO(3)}{=} |\alpha'|^2 |\alpha''|^2 - |\langle \alpha', \alpha'' \rangle|^2 = |\alpha' \times \alpha''|^2.$$

$$\langle \bar{\alpha}' \times \bar{\alpha}'', \bar{\alpha}''' \rangle = \langle A\alpha' \times A\alpha'', A\alpha''' \rangle$$

column. vectors

$$= \det \left[A\alpha' \; A\alpha'' \; A\alpha''' \right] \qquad \begin{bmatrix} | & | & | \\ A\alpha' & A\alpha'' & A\alpha''' \\ | & | & | \end{bmatrix}$$

matrix multiplication

$$= \det \left(A [\alpha' \; \alpha'' \; \alpha'''] \right)$$

$A \in SO(3) = \cancel{\det} A' \det[\alpha' \ \alpha'' \ \alpha''']$ $\quad = A \begin{bmatrix} | & | & | \\ \alpha' & \alpha'' & \alpha''' \\ | & | & | \end{bmatrix}$.

$= \langle \alpha' \times \alpha'', \alpha''' \rangle$.

$\Rightarrow \overline{\tau}(s) = \tau(s)$.

So we have shown that for $\alpha(s)$, $A \in SO(3)$, $p \in \mathbb{R}^3$

$\overline{\alpha}(s) = A \cdot \alpha(s) + p$ will have same arc-length, curvature, torsion as $\alpha(s)$

Pf of Thm (Uniqueness Part):

Note: Existence part will be covered later. – ODE theory

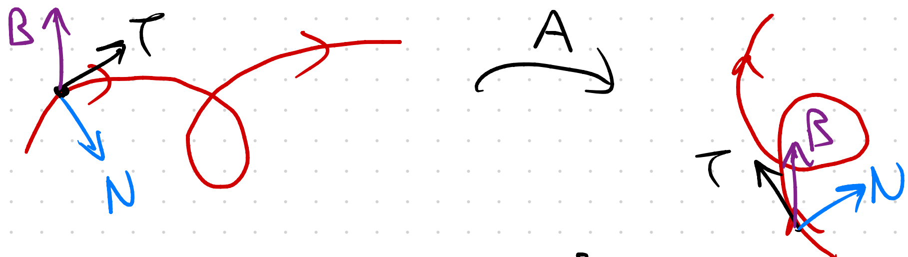
Given $\alpha(s)$, $\overline{\alpha}(s)$, $s \in I$ s.t. $k(s) = \overline{k}(s)$

$\tau(s) = \overline{\tau}(s)$.

WTS $\exists A \in SO(3)$, $x_0 \in \mathbb{R}^3$ s.t. $\overline{\alpha}(s) = A \cdot \alpha(s) + x_0$.

First want to find A:

idea: look at the Frénet frames at a fixed point.



i.e. rotation A is determined by $\{T, N, B\}(s_0) \to \{\overline{T}, \overline{N}, \overline{B}\}(s_0)$.

So given $s_0 \in I$, $\exists\, A = A(s_0) \in SO(3)$ s.t.

$$\begin{cases} AT(s_0) = \overline{T}(s_0) \\ AN(s_0) = \overline{N}(s_0) \\ AB(s_0) = \overline{B}(s_0) \end{cases}$$

$$\left(\overline{B} = \overline{T} \times \overline{N} = AT \times AN \underset{\substack{\uparrow \\ A \in SO(3)}}{=} A(T \times N) = AB, \right.$$ quicker way: $SO(3)$ matrices preserve O.N. frames$\Big)$.

Now it remains to show $\bar{\alpha} - A\alpha \equiv$ const. in s. (i.e. only translation left).

WLOG, by replacing α with $A\alpha$, I can assume $A = Id$.

$$\frac{1}{2}\frac{d}{ds}|T-\bar{T}|^2 = \frac{1}{2}\frac{d}{ds}\langle T-\bar{T}, T-\bar{T}\rangle = \langle T-\bar{T}, T'-\bar{T}'\rangle \qquad ' \text{ is wrt. } s$$

$$\overset{\text{Frénet formula}}{=} \langle T-\bar{T}, kN - \bar{k}\bar{N}\rangle$$

$$\overset{k=\bar{k}}{=} k\langle T-\bar{T}, N-\bar{N}\rangle.$$

$$\frac{1}{2}\frac{d}{ds}|N-\bar{N}|^2 = \langle N-\bar{N}, N'-\bar{N}'\rangle$$

$$= \langle N-\bar{N}, -kT+\tau B+\bar{k}\bar{T}-\bar{\tau}\bar{B}\rangle$$

$$= -k\langle N-\bar{N}, T-\bar{T}\rangle$$

$$+ \tau\langle N-\bar{N}, B-\bar{B}\rangle$$

$$\frac{1}{2}\frac{d}{ds}|B-\bar{B}|^2 = \langle B-\bar{B}, B'-\bar{B}'\rangle$$

$$= -\tau\langle B-\bar{B}, N-\bar{N}\rangle.$$

So $\frac{d}{ds}\left(|T-\overline{T}|^2+|N-\overline{N}|^2+|B-\overline{B}|^2\right)$ (e.g. $\langle N-\overline{N}, B-\overline{B}\rangle \leq C\left(|N-\overline{N}|^2 + |B-\overline{B}|^2\right)$) Cauchy

Cauchy $\leq C \sup_I (|\kappa|+|\tau|)\left(|T-\overline{T}|^2+|N-\overline{N}|^2+|B-\overline{B}|^2\right)$

$$\begin{cases} f' \leq Cf & \text{on } I. \\ f(s_0)=0 \end{cases} \Rightarrow f(s)=0 \quad \forall s\in[s_0,b].$$

Here $f := |T-\overline{T}|^2 + |N-\overline{N}|^2 + |B-\overline{B}|^2$.

Reversing the orientation and repeating the argument yields $f(s)=0 \ \forall s\in[a,s_0)$.

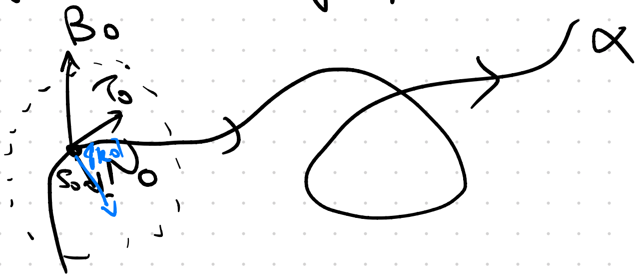
$\Rightarrow f(s)\equiv 0$ on I.

$$\Rightarrow \begin{cases} T=\overline{T} \\ N=\overline{N} \\ B=\overline{B} \end{cases} \text{on } I. \quad \Rightarrow (\alpha-\overline{\alpha})' = (T-\overline{T}) = 0.$$

So $\overline{\alpha} = \alpha + \text{const}$.

Local expression of α:

given arc-length param. curve $\alpha : I \to \mathbb{R}^3$, fixed $s_0 \in I$ (WLOG, $s_0 = 0$)



By Taylor expansion:

$$\alpha(s) = \alpha(0) + s\alpha'(0) + \frac{1}{2}s^2\alpha''(0) + \frac{1}{6}s^3\alpha'''(0) + o(s^3).$$

$$= \alpha(0) + sT_0 + \frac{1}{2}s^2 k_0 N_0 + \frac{1}{6}s^3\left(k_0' N_0 - k_0^2 T_0 + k_0 \tau_0 B_0\right) + o(s^3).$$

$$= \alpha(0) + \left(s - \frac{1}{6}s^3 k_0^2\right)T_0 + \left(\frac{1}{2}s^2 k_0 + \frac{1}{6}s^3 k_0'\right)N_0$$

$$+ \frac{1}{6}s^3 k_0 \tau_0 B_0 + o(s^3).$$

Meaning : at $\alpha(0) \in \mathbb{R}^3$,

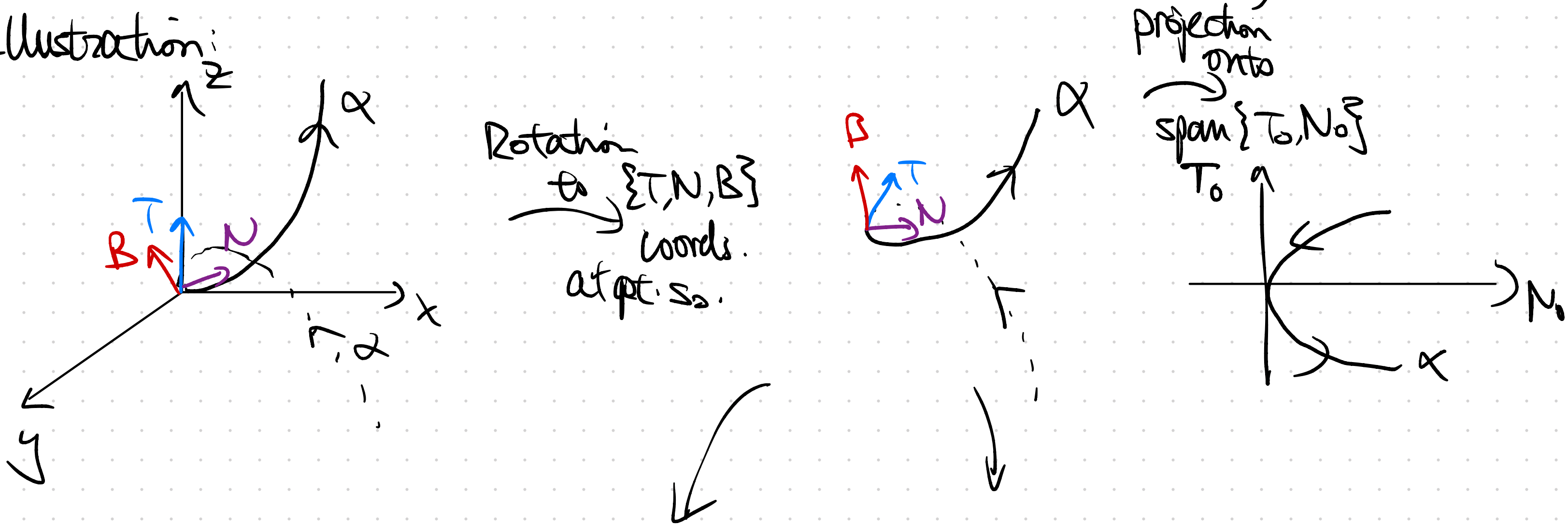
instead of considering xyz coords ($\operatorname{span}\{\partial_x, \partial_y, \partial_z\}$)

if we consider coordinates induced by $\{T_0, N_0, B_0\}$

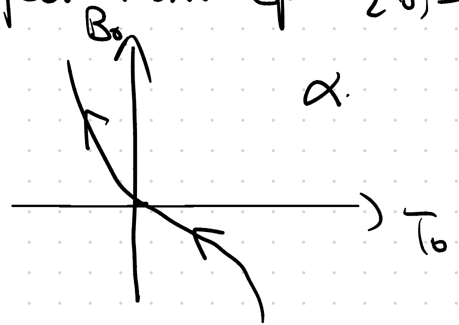
Then we can write

$$\alpha(s) = \left(s - \frac{1}{6}s^3 k_0^2,\ \frac{1}{2}s^2 k_0 + \frac{1}{6}s^3 k_0',\ \frac{1}{6}s^3 k_0 \tau_0\right) + o(s^3)$$

Illustration:



projection onto span $\{T_0, B_0\}$



projection onto $\{N_0, B_0\}$